

# Math 246A Lecture 27 Notes

Daniel Raban

November 30, 2018

## 1 Schwartz's Reflection Principle and Harnack's Principle

### 1.1 Schwartz's reflection principle

**Lemma 1.1** (Schwartz's reflection principle). *Let  $v(z)$  be harmonic in  $\mathbb{D} \cap \{z : y > 0\}$ , and suppose that  $\lim_{y \rightarrow 0} v(x + iy) = 0$ . Then  $v$  extends to be harmonic on all of  $\mathbb{D}$  by reflection:*

$$V(z) = \begin{cases} v(z) & \text{Im}(z) > 0 \\ 0 & \text{Im}(z) = 0 \\ -v(\bar{z}) & \text{Im}(z) < 0 \end{cases}$$

*is harmonic on  $\mathbb{D}$ .*

*Proof.* The mean value property implies  $V$  is harmonic; for points on the boundary, the contribution of the upper and lower arcs to the integral cancel out.  $\square$

**Definition 1.1.** Let  $\Omega$  be a domain in  $\mathbb{C}$  and let  $\gamma \subseteq \partial\Omega$ .  $\gamma$  is a **free boundary arc** if

1.  $\gamma = \{\psi(t) : t \in (0, 1)\}$ , where  $\psi : \mathbb{D} \rightarrow \Omega$  is a homeomorphism.
2.  $\gamma \subseteq \partial\Omega$ , and

$$p \in \gamma \implies \exists \delta > 0 \text{ s.t. } B(p, \delta) \cap \partial\Omega = \gamma \cap B(p, \delta).$$

**Definition 1.2.**  $\gamma$  is an **analytic arc** if for open  $U \subseteq \mathbb{C}$  with  $(0, 1) \subseteq U$ ,  $\psi : U \rightarrow \mathbb{C}$ ,  $\gamma = \psi(0, 1)$ , where  $\psi$  is analytic, 1-1 and onto  $\psi(U)$ .

Assume  $\gamma$  is a free boundary arc on a domain  $\Omega$ . Now split  $U = U^+ \cup U^- \cup (0, 1)$ , where  $U^\pm$  are connected components of  $U \setminus (0, 1)$ .

If  $\psi(U) \cap \Omega$  is  $\psi(U^+)$  or  $\psi(U^-)$ , we say that  $\gamma$  is a **one-sided** analytic boundary arc. Otherwise, if  $\psi(U) \cap \Omega = \psi(U^+) \cup \psi(U^-)$ , we say that  $\gamma$  is a **two-sided** analytic boundary arc.

**Definition 1.3.** Let  $\varphi : \Omega \rightarrow \Omega'$ . Then  $\varphi$  is **proper** if for  $z_n \rightarrow \partial\Omega$ ,  $\varphi(z_n) \rightarrow \partial\Omega'$ .

**Lemma 1.2.** *Suppose  $\varphi : \Omega \rightarrow \Omega$  is a homeomorphism. Then  $\varphi$  is proper.*

*Proof.* If not, for some subsequence,  $\varphi(z_{n_j}) \rightarrow w \in \Omega$ . Then  $z_{n_j} \rightarrow \zeta \in U^{-1}(U)$ . □

**Theorem 1.1.** *Assume  $\Omega$  is a simply connected domain and  $\gamma$  is a one-sided analytic arc on  $\partial\Omega$ . Let  $U, \psi(U)$  be as above. Let  $\varphi : \Omega \rightarrow \mathbb{D}$  be conformal. Then  $\varphi$  extends to a 1-1 holomorphic map  $\tilde{\varphi} : \Omega \cup \psi(U) \rightarrow \mathbb{C}$  such that  $|\tilde{\varphi}(z)| = 1$  on  $\gamma$ , and*

$$|\tilde{\varphi}(z)| > 1 \iff z \in \psi(U) \setminus \overline{\Omega}.$$

*Proof.* Without loss of generality,  $\Omega \cap \psi(U) = \psi(U^+)$  and  $\varphi^{-1}(0) \notin \psi(U)$ . Consider  $v(u) = -\log |\varphi(\psi(z))|$  for  $z \in U^+$ . Then  $v(z) \rightarrow 0$  as  $z \rightarrow (0, 1)$  since  $\varphi$  is proper. So there exists  $V$  harmonic on  $U$ , and  $V = \text{Re}(F)$ , where  $F \in H(U)$ . We can say that  $F(z_0) = e^{\varphi \circ \psi(z_0)}$  for some  $z_0 \in U^+$  (since  $F$  is defined up to a constant). Let

$$\tilde{\varphi} = \begin{cases} \varphi & \text{on } \Omega \\ F \circ \psi^{-1} & \text{on } \psi(U). \end{cases}$$

Then  $\tilde{\varphi}$  is analytic and has the desired properties. □

## 1.2 Harnack's principle

**Lemma 1.3.** *Let  $u$  be a positive harmonic function on an open set  $U \supseteq \overline{B(z_0, R)}$ . Let  $|z - z_0| = r < R$ . Then*

$$\frac{R-r}{R+r}u(z_0) \leq u(z) \leq \frac{R+r}{R-r}u(z_0).$$

*Proof.* By a change of variables and the Poisson integral formula,

$$u(z) = \frac{1}{2\pi} \int \frac{R^2 - r^2}{|Re^{i\theta} - (z - z_0)|^2} u(z_0 + Re^{i\theta}) d\theta.$$

Use the M-L estimate. The maximum and the minimum of the kernel  $(R^2 - r^2)/|Re^{i\theta} - (z - z_0)|^2$  are  $(R+r)/(R-r)$  and  $(R-r)/(R+r)$ . □

**Theorem 1.2** (Harnack's principle). *Let  $\Omega_n$  be a sequence of domains and let  $u_n : \Omega_n \rightarrow \mathbb{R}$  be harmonic. Let  $\Omega$  also be a domain. Assume that:*

1. *If  $K \subseteq \Omega$  is compact, then there exists  $n_K$  such that  $K \subseteq \Omega_n$  for  $n > n_K$ .*
2. *If  $K \subseteq \Omega$  is compact, there exists  $n_1(K)$  such that if  $n > n_1(K)$ , then  $u_{n+1} \geq u_n$  on  $K$ .*

*Then either*

1.  *$u_n \rightarrow +\infty$  uniformly on  $K$  for all compact  $K \subseteq \Omega$ .*
2. *there exists a harmonic function  $u$  on  $\Omega$  such that  $u_n \rightarrow u$  uniformly on all compact  $K \subseteq \Omega$ .*